8.

Differentiation techniques and applications

- Explicitly defined and implicitly defined functions
- Differentiation of implicitly defined functions
- Differentiation of parametrically defined functions
- Related rates
- Small changes
- Marginal rates of change
- Logarithmic differentiation
- Miscellaneous exercise eight

Explicitly defined and implicitly defined functions

When a function is expressed in the form $y = f(x)$, the relationship that exists between the two variables is clearly defined with one variable isolated on one side of the equation and all other terms, containing only the other variable, on the other side. For example:

$$y = 2x + 3, \qquad y = \frac{x-3}{x+5}, \qquad y = 2x^3 + 7x^2 - 3x + 4.$$

In this form, the functions are defined **explicitly**.

Consider instead a function defined by the equation $xy + y - 4x = -2$. In this form the relationship between x and y is implied, but is not explicitly set out. The function is said to be defined **implicitly**.

Differentiation of implicitly defined functions

Implicit differentiation

Suppose we want to find the gradient at a particular point on the graph of an implicitly defined function.

For example, suppose we want to determine the gradient of

$$xy + y - 4x = -2$$

at the point (5, 3).

For this particular example we could rearrange the given expression into an explicitly defined form and then differentiate as usual:

If $\quad xy + y - 4x = -2$

then $\quad y(x+1) = 4x - 2$

$$y = \frac{4x-2}{x+1}$$

Hence $\quad \dfrac{dy}{dx} = \dfrac{(x+1)(4) - (4x-2)(1)}{(x+1)^2}$

At (5, 3) $\quad \dfrac{dy}{dx} = \dfrac{(6)(4) - (18)(1)}{36}$

$$\frac{dy}{dx} = \frac{1}{6}$$

$$\frac{d}{dx}\left(\frac{4x-2}{x+1}\right) | x = 5$$

$$\frac{1}{6}$$

However it is possible to differentiate the implicitly defined function

$$xy + y - 4x = -2$$

directly, without first isolating y, as shown on the next page.

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$$
\begin{aligned}
\text{Given:} \quad && xy + y - 4x &= -2 \\
\text{Differentiate with respect to } x: \quad && \frac{d}{dx}(xy) + \frac{d}{dx}(y) - \frac{d}{dx}(4x) &= \frac{d}{dx}(-2) \\
&& y\frac{d}{dx}(x) + x\frac{d}{dx}(y) + \frac{dy}{dx} - 4 &= 0 \\
&& y(1) + x\frac{dy}{dx} + \frac{dy}{dx} - 4 &= 0 \\
\text{At } (5, 3) \quad && 3 + 5\frac{dy}{dx} + \frac{dy}{dx} - 4 &= 0 \\
&& 6\frac{dy}{dx} &= 1 \\
\therefore \quad && \frac{dy}{dx} &= \frac{1}{6}, \quad \text{as before.}
\end{aligned}
$$

This second method may not be any quicker than the first but in some cases the task of isolating y may be difficult and perhaps even impossible. This would deny the use of the first method but the second method would still work.

This second method is further demonstrated in the examples that follow.

In these examples note especially the use of the rearrangement

$$\frac{d}{dx}f(y) = \frac{d}{dy}f(y)\frac{dy}{dx} \qquad \text{(i.e. the chain rule).}$$

Note also that whilst the implicitly defined equations may represent a relationship that is not a function, $\frac{dy}{dx}$ can still be determined.

EXAMPLE 1

Find $\frac{dy}{dx}$ in terms of x and y if $y^2 + 5x = 6x^2y$.

Solution

$$
\begin{aligned}
\text{Given:} \quad && y^2 + 5x &= 6x^2y \\
\text{Differentiate with respect to } x: \quad && \frac{d}{dx}(y^2) + \frac{d}{dx}(5x) &= \frac{d}{dx}(6x^2y) \\
&& \frac{d}{dy}(y^2)\frac{dy}{dx} + 5 &= y\frac{d}{dx}(6x^2) + 6x^2\frac{d}{dx}(y) \\
&& 2y\frac{dy}{dx} + 5 &= 12xy + 6x^2\frac{dy}{dx} \\
\therefore \quad && \frac{dy}{dx}(2y - 6x^2) &= 12xy - 5 \\
\text{Thus} \quad && \frac{dy}{dx} &= \frac{12xy - 5}{2y - 6x^2}
\end{aligned}
$$

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Alternatively, using a calculator:

```
impDiff(y^2 + 5x = 6x^2 y, x, y)
```

$$y' = \frac{-(12 \cdot x \cdot y - 5)}{2 \cdot (3 \cdot x^2 - y)}$$

EXAMPLE 2

Determine the gradient of the curve $3y^2 + 4x = x^2 + 2xy - 8$ at the point (5, 3).

Solution

Given: $3y^2 + 4x = x^2 + 2xy - 8$

Differentiate with respect to x:

$$\frac{d}{dx}(3y^2) + \frac{d}{dx}(4x) = \frac{d}{dx}(x^2) + \frac{d}{dx}(2xy) - \frac{d}{dx}(8)$$

$$\frac{d}{dy}(3y^2)\frac{dy}{dx} + 4 = 2x + y\frac{d}{dx}(2x) + 2x\frac{d}{dx}(y)$$

$$6y\frac{dy}{dx} + 4 = 2x + 2y + 2x\frac{dy}{dx}$$

At (5, 3)

$$18\frac{dy}{dx} + 4 = 10 + 6 + 10\frac{dy}{dx}$$

$\therefore$

$$\frac{dy}{dx} = \frac{3}{2}$$

At (5, 3) the curve has gradient $\frac{3}{2}$.

```
impDif(3·y^2 + 4·x = x^2 + 2·x·y − 8, x, y) | x = 5
```

$$\frac{y+3}{3 \cdot y - 5}$$

$$\frac{y+3}{3 \cdot y - 5} \mid y = 3$$

$$\frac{3}{2}$$

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EXAMPLE 3

If $y^2 + x = x^3 - y + 6$ determine $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in terms of x and y and evaluate each of these for the point (1, 2).

Solution

Given

$$y^2 + x = x^3 - y + 6$$

Differentiate with respect to x:

$$\begin{aligned}
\frac{d}{dx}(y^2) + \frac{d}{dx}(x) &= \frac{d}{dx}(x^3) - \frac{d}{dx}(y) + \frac{d}{dx}(6) \\
\frac{d}{dy}(y^2)\frac{dy}{dx} + 1 &= 3x^2 - \frac{dy}{dx} \\
2y\frac{dy}{dx} + 1 &= 3x^2 - \frac{dy}{dx} \\
\frac{dy}{dx}(2y+1) &= 3x^2 - 1 \\
\frac{dy}{dx} &= \frac{3x^2 - 1}{2y+1}
\end{aligned}$$

At the point (1, 2)

$$\begin{aligned}
\frac{dy}{dx} &= \frac{3(1)^2 - 1}{2(2)+1} \\
&= \frac{2}{5}
\end{aligned}$$

$\frac{d^2y}{dx^2}$ is the second derivative of y with respect to x.

Thus

$$\begin{aligned}
\frac{d^2y}{dx^2} &= \frac{d}{dx}\left(\frac{dy}{dx}\right) \\
&= \frac{d}{dx}\left(\frac{3x^2 - 1}{2y+1}\right) \\
&= \frac{(2y+1)6x - (3x^2-1)2\frac{dy}{dx}}{(2y+1)^2} \\
&= \frac{(2y+1)6x - (3x^2-1)2\frac{3x^2-1}{2y+1}}{(2y+1)^2} \\
&= \frac{6x(2y+1)^2 - 2(3x^2-1)^2}{(2y+1)^3}
\end{aligned}$$

Investigate the ability of your calculator to determine the second derivative of implicitly defined functions.

At the point (1, 2)

$$\begin{aligned}
\frac{d^2y}{dx^2} &= \frac{6(1)(5)^2 - 2(2)^2}{(5)^3} \\
&= \frac{142}{125}
\end{aligned}$$

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Exercise 8A

Without first rearranging the equation, differentiate each of the following with respect to x to find $\frac{dy}{dx}$ in terms of x and y.

1 $xy + 8x = 10 - 2y$

2 $xy + y - 4x = 3x^2 - 5$

3 $y^3 - 2x = 3x^2y$

4 $y^2 = 2x^3y + 5x$

5 $5y^2 = x^2 + 2xy - 3x$

6 $x + 3y^2 = 5 + x^2 + 2xy$

7 $x^2 + y^2 = 9x$

8 $x^2 + y^2 = 9y$

9 $x^2 + y^2 = 9xy$

10 $x^2 + y^2 = 9xy + x + y$

11 $\sin x + \cos y = 10$

12 $3 + x^2 \cos y = 10xy$

13 Determine the gradient of $6x + xy + 20 + 2y = 0$ at the point $(-3, 2)$.

14 Determine the gradient of $6y + xy = 10 + 3x$ at the point $(2, 2)$.

15 Determine the gradient of $5 + x^3 = xy + y^2$ at the point $(1, -3)$.

16 Determine the gradient of $y^2 + 3xy = 4x$ at the point $(1, -4)$.

17 Find the equation of the tangent to $x^2 + \frac{y}{x} = 2y$ at the point $(1, 1)$.

18 Determine the gradient of $5x^2 + \sqrt{xy} = 5 + y^2$ at the point $(4, 9)$.

19 If $\frac{dy}{dx} = x^2y$ find an expression for $\frac{d^2y}{dx^2}$ in terms of x and y.

20 Determine the coordinates of the points on the graph of $x^2 + 4y^2 - 2x + 6y = 17$ where the tangent to the curve is horizontal.

21 Determine the coordinates of the points on the graph of $x^2 + y^2 - 4x + 6y + 12 = 0$ where the tangent to the curve is vertical.

22 If $y - y^3 = x^2 + x - 2$ determine $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in terms of x and y and evaluate each of these for the point $(1, 0)$.

23 Find the equation of the tangent to the curve $x^2 = 2\sin y$ at the point $\left(1, \frac{\pi}{6}\right)$.

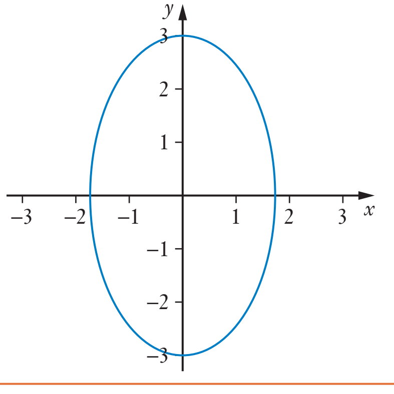
24 If $y^2 + \cos x = 3y + 1$ determine $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in terms of x and y.

25 If $2\sin y - x^2 = 2x + 1$ determine $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in terms of x and y and evaluate each of these for the point $\left(-2, \frac{\pi}{6}\right)$.

26 The graph on the right shows the ellipse with equation

$$3x^2 + y^2 = 9$$

Find the coordinates of any points on the ellipse where the gradient is -1.



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Differentiation of parametrically defined functions

Note

This topic is not specifically mentioned in the syllabus for this unit but I include it here because it is only bringing together the idea of defining a function parametrically, as encountered in Unit Three of *Mathematics Specialist*, with our ability to use the chain rule.

Consider the parametric equations $x = 3t + 1$ and $y = t^2$.

In this case we can eliminate t to express y directly in terms of x.

From the first equation: $t = \frac{x-1}{3}$

Substituting into the second equation: $y = \frac{(x-1)^2}{9}$

From which $\frac{dy}{dx} = \frac{2(x-1)}{9}$.

If we require the gradient at a particular point on the curve, say where $t = 1$, i.e. the point (4, 1), then

$$\frac{dy}{dx} = \frac{2}{3}.$$

However, use of the chain rule allows this derivative to be determined without first having to eliminate the parameter. (Particularly useful for those cases in which expressing y directly in terms of x is difficult or even impossible.)

If $x = 3t + 1$ and $y = t^2$

then $\frac{dx}{dt} = 3$ and $\frac{dy}{dt} = 2t$.

Now use the chain rule:

$$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx}$$

$$= (2t)\left(\frac{1}{3}\right)$$

If $t = 1$ $\frac{dy}{dx} = \frac{2}{3}$ as before.

Note: • The above example used the fact that: $\frac{dt}{dx} = \frac{1}{\left(\frac{dx}{dt}\right)}$.

We cannot simply assume this result to be true by the rules of fractions because $\frac{dt}{dx}$ is not a fraction (it is the limit of a fraction). Instead the result can be justified as follows:

Using the chain rule: $\frac{dz}{dt}\frac{dt}{dx} = \frac{dz}{dx}$ [1]

Now suppose that $z = x$. Differentiation gives $\frac{dz}{dx} = 1$.

Equation [1] then becomes $\frac{dx}{dt}\frac{dt}{dx} = 1$ and so $\frac{dt}{dx} = \frac{1}{\left(\frac{dx}{dt}\right)}$ as required.

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- As was mentioned in Unit Three of *Mathematics Specialist*, some graphic calculators can accept and display relationships defined parametrically. Use such a calculator to display the graph defined parametrically as:

$$\begin{cases} x = \sin t \\ y = 2\cos 3t \end{cases}.$$

Exercise 8B

1 If $x = 3\sin 2t$ and $y = 2\cos 5t$ find, in terms of t,

a $\frac{dx}{dt}$ **b** $\frac{dy}{dt}$ **c** $\frac{dy}{dx}$.

2 If $x = \sin^2 t$ and $y = \cos 3t$ find, in terms of t,

a $\frac{dx}{dt}$ **b** $\frac{dy}{dt}$ **c** $\frac{dy}{dx}$.

Find $\frac{dy}{dx}$ in terms of the parameter t for each of the following.

3 $x = 2 + 3t, y = t^2$.

4 $x = t^2, y = 2 + 3t$.

5 $x = 5t^3, y = t^2 + 2t$.

6 $x = 3t^2 + 6t, y = \frac{1}{t+1}$.

7 $x = t^2 - 1, y = (t-1)^2$.

8 $x = \frac{t}{t-1}, y = \frac{2}{t+1}$.

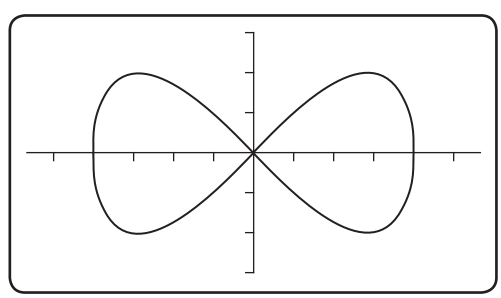
9 Determine the gradient at the point where $t = -1$, for the curve that is defined parametrically by $x = t^2 + 2, y = t^3$.

10 Determine the gradient at the point where $t = 2$, for the curve that is defined parametrically by $x = \frac{1}{t+1}, y = t^2 + 1$.

11 Determine the coordinates, (x, y), of any points where $\frac{dy}{dx} = 0$ on the curve defined parametrically by $x = 2t^2 + 3t, y = t^3 - 12t$.

12 The diagram on the right shows the curve defined parametrically as:

$x = 4\sin t,$ $y = 2\sin 2t,$ for $0 \le t \le 2\pi$.



a Find an expression for $\frac{dy}{dx}$ in terms of t.

b Find the coordinates and the gradient at the point where $t = \frac{\pi}{6}$.

c Find the exact values of t (for $0 \le t \le 2\pi$) for which $\frac{dy}{dx} = 0$.

13 If $y = t + \frac{2}{t}$ and $x = 2t - \frac{1}{t}$ find **a** $\frac{dy}{dx}$ in terms of t, **b** $\frac{d^2y}{dx^2}$ in terms of t.

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WS
Related rates

Related rates

Suppose we know the rate of change of one variable, say x, with respect to some second variable, say t. If we also know the relationship between x and some third variable, say y, then we are able to determine the rate of change of y, with respect to t.

i.e. Knowing $\frac{dx}{dt}$ and y as a function of x, we can determine $\frac{dy}{dt}$.

For example:

Suppose $\frac{dx}{dt} = 5$ and $y = 5x^2 + 2x - 3$.

$\frac{dy}{dt}$ can be obtained as follows:

If $y = 5x^2 + 2x - 3$

then $\frac{dy}{dx} = 10x + 2$.

By the chain rule
$$\begin{aligned}\frac{dy}{dt} &= \frac{dy}{dx}\frac{dx}{dt}\\ &= (10x+2)5\\ &= 50x+10\end{aligned}$$

Alternative setting out:
$$\frac{dy}{dt} = \frac{d}{dt}(5x^2 + 2x - 3)$$

By the chain rule
$$\begin{aligned}&= \frac{d}{dx}(5x^2+2x-3)\frac{dx}{dt}\\ &= (10x+2)5\\ &= 50x+10\end{aligned}$$

EXAMPLE 4

If $A = 3w^2$ and $\frac{dw}{dt} = 5$ find $\frac{dA}{dt}$ when $w = 2$.

Solution

If $A = 3w^2$ then
$$\begin{aligned}\frac{dA}{dt} &= \frac{d}{dt}(3w^2)\\ &= \frac{d}{dw}(3w^2)\frac{dw}{dt}\\ &= (6w)(5)\\ &= 30\,w\\ &= 60 \qquad \text{when } w = 2.\end{aligned}$$

Thus when $w = 2$, $\frac{dA}{dt}$ is 60.

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EXAMPLE 5

If $y^2 = 1.5w^3 + 2.5$ and $\frac{dw}{dt} = 12$, find $\frac{dy}{dt}$ when $y = 2$.

Solution

If $\quad y^2 = 1.5w^3 + 2.5 \quad$ then

$$\frac{d}{dt}(y^2) = \frac{d}{dt}(1.5w^3 + 2.5)$$

$$\frac{d}{dy}(y^2)\frac{dy}{dt} = \frac{d}{dw}(1.5w^3 + 2.5)\frac{dw}{dt}$$

$$2y\frac{dy}{dt} = (4.5w^2)(12)$$

$$\therefore \quad \frac{dy}{dt} = \frac{27w^2}{y}$$

Now when $\quad y = 2$

$4 = 1.5w^3 + 2.5$

giving $\quad w = 1.$

Thus when $\quad y = 2, \quad w = 1 \quad$ and $\quad \frac{dy}{dt} = \frac{27}{2}$

Thus when $y = 2$, $\frac{dy}{dt}$ is 13.5.

EXAMPLE 6

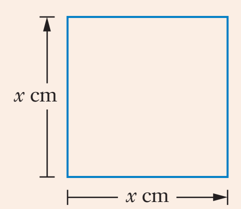
The length of each side of a square is increased at a rate of 2 mm/s. Find the rate of increase in the area of the square when the side length is 10 cm.

Solution

Note: When the question does not specifically state what the rate is with respect to, as in this question, we assume it to be with respect to *time*.

Draw a diagram showing the situation at some general time t:

In the diagram shown, x cm is the side length at time t seconds.



Thus we are given that $\quad \frac{dx}{dt} = 0.2$

If the area at time t is A cm^2 then $\quad A = x^2$

Differentiating with respect to t:

$$\frac{dA}{dt} = \frac{d}{dt}(x^2)$$

$$= \frac{d}{dx}(x^2)\frac{dx}{dt}$$

$$= (2x)(0.2)$$

$$= 4 \qquad \text{when } x = 10.$$

When the side length is 10 cm, the area of the square is increasing at 4 cm^2/s.

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EXAMPLE 7

A particle moves in a straight line such that its velocity, v m/s, depends upon the particle's displacement, x m, from some fixed point O according to the rule

$$v = 3x + 4$$

Find the velocity and acceleration of the particle when $x = 1$.

Solution

When $x = 1$

$$\begin{aligned} v &= 3(1) + 4 \\ &= 7 \end{aligned}$$

When $x = 1$, the velocity is 7 m/s.

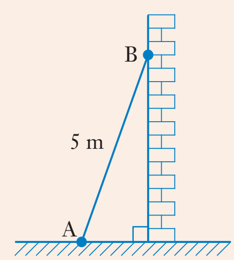
The acceleration is

$$\begin{aligned} \frac{dv}{dt} &= \frac{d}{dt}(3x+4) \\ &= \frac{d}{dx}(3x+4)\frac{dx}{dt} \\ &= (3)(v) \\ &= (3)(7) \qquad \text{when } x = 1. \\ &= 21 \end{aligned}$$

When $x = 1$ the acceleration is 21 m/s^2.

EXAMPLE 8

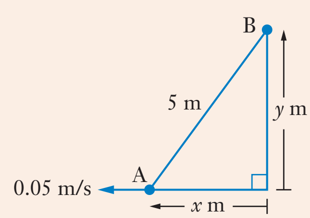
The diagram shows a ladder AB, of length 5 metres, with its foot on horizontal ground and its top leaning against a vertical wall.



End A slips along the floor, directly away from the wall, at a speed of 0.05 m/s. How fast is end B moving down the wall at the instant that it passes through the point that is 4 m above the ground?

Solution

Draw a diagram showing the situation at some general time t:



With x and y as in the diagram $\quad x^2 + y^2 = 25.$

Thus

$$\begin{aligned} \frac{d}{dt}(x^2+y^2) &= \frac{d}{dt}(25) \\ 2x\frac{dx}{dt} + 2y\frac{dy}{dt} &= 0 \\ \therefore \quad y\frac{dy}{dt} &= -x\frac{dx}{dt} \end{aligned}$$

From $x^2 + y^2 = 25$, when $y = 4$, $x = 3$. ($x = -3$ not possible in this situation).

Thus when $y = 4$

$$\begin{aligned} 4\frac{dy}{dt} &= (-3)(0.05) \\ \therefore \quad \frac{dy}{dt} &= -0.0375 \end{aligned}$$

When B is 4 m above the ground it is moving down the wall at 0.0375 m/s.

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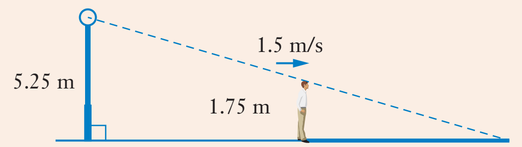
EXAMPLE 9

A person who is 1.75 m tall walks directly away from a light source positioned 5.25 m above ground. If the person walks at a steady 1.5 m/s find

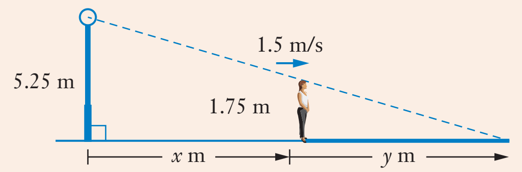


a how fast the length of the person's shadow is changing,

b how fast the tip of the person's shadow is moving across the ground.

Solution

a Draw a diagram showing the situation at some general time t:



With x and y as in the diagram we are given $\frac{dx}{dt} = 1.5$ and we require $\frac{dy}{dt}$.

By similar triangles $\quad \frac{5.25}{1.75} = \frac{x+y}{y}$.

Which simplifies to $\quad 2y = x$

Differentiate with respect to t:

$$\frac{d}{dt}(2y) = \frac{d}{dt}(x)$$

$$\frac{d}{dy}(2y)\frac{dy}{dt} = \frac{dx}{dt}$$

$$2\frac{dy}{dt} = \frac{dx}{dt}$$

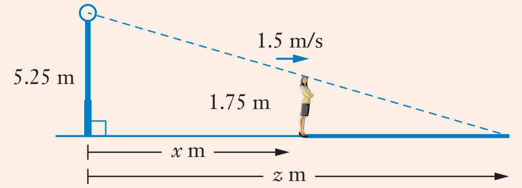
Thus with $\frac{dx}{dt} = 1.5$, $\quad \frac{dy}{dt} = 0.75$

The length of the person's shadow is changing at 0.75 m/s.

b The rate of change of y found in **a** will not be the rate at which the tip of the shadow is moving across the ground because we are measuring y from a moving point, the position of the person. The person is moving at 1.5 m/s and the shadow is lengthening at 0.75 m/s. Thus the tip of the shadow is moving at (1.5 + 0.75) m/s.

Thus the tip of the shadow is moving across the ground at 2.25 m/s.

Alternatively we could consider the motion of the tip of the shadow by considering a variable that is measured from some fixed point. In the diagram on the right, z m is the distance from the base of the lamp post to the tip of the shadow.



We are given $\frac{dx}{dt} = 1.5$ and we require $\frac{dz}{dt}$.

By similar triangles $\quad \frac{5.25}{1.75} = \frac{z}{z-x}$.

Which simplifies to $\quad 2z = 3x$

Hence $\quad \frac{dz}{dt} = 1.5\frac{dx}{dt}$

$\therefore \quad = 1.5\,(1.5) = 2.25$ as before.

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Exercise 8C

1 If $y = 3x^2 + 4x$ and $\frac{dx}{dt} = 5$, find $\frac{dy}{dt}$ when $x = 6$.

2 If $A = 8p^3$ and $\frac{dp}{dt} = 0.25$, find $\frac{dA}{dt}$ when $p = 0.5$.

3 If $X = \sin 2p$ and $\frac{dp}{dt} = 2$, find $\frac{dX}{dt}$ when $p = \frac{\pi}{6}$.

4 If $T = \frac{2\pi}{3}\sqrt{L}$ find

a $\frac{dT}{dt}$ given that $\frac{dL}{dt} = \frac{15}{\pi}$ and $L = 100$,

b $\frac{dL}{dt}$ given that $\frac{dT}{dt} = 6\pi$ and $L = 4$.

5 If $A = \sin^2(3x)$ and $\frac{dx}{dt} = 0.1$, find $\frac{dA}{dt}$ when $x = \frac{\pi}{36}$.

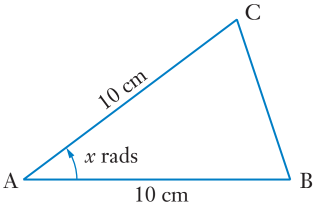
6 If $P = 4r^2 + 3$ and $\frac{dP}{dt} = 14$, find $\frac{dr}{dt}$ when $r = 7$.

7 If $y^2 = 3x^3 + 1$ and $\frac{dx}{dt} = 0.1$, find $\frac{dy}{dt}$ when $y = 5$.

8 If $x^2 + y^2 = 400$, $x \geq 0$ and $\frac{dx}{dt} = 6$, find $\frac{dy}{dt}$ when $y = 12$.

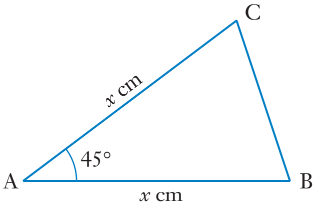
9 The isosceles triangle on the right has two sides of length 10 cm and ∠CAB is increasing at 0.01 radians/second, from an initial $\frac{\pi}{6}$ radians.

Find the rate of change in the area of ΔABC at the instant when $x = \frac{\pi}{3}$.



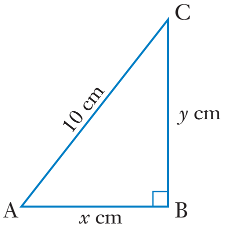
10 The isosceles triangle on the right has sides AC and AB of equal length and each increasing at 0.1 cm/s from an initial 5 cm.

Find the rate of change in the area of ΔABC at the instant when AC = AB = 10 cm.



11 The right triangle shown on the right has sides AB and BC able to vary in length, AC is constantly 10 cm long and ∠ABC is always a right angle. Initially AB is of length 4 cm and increases at a constant rate of 0.1 cm/s.

Find the rate of change in the length of BC, 20 seconds after the increasing in the length of AB commenced.



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12 The length of each side of a square is increased at a rate of 0.01 cm/s. Find the rate at which the area of the square is increasing when the side length is 8 cm.

13 The length of a particular rectangle is three times its width and this ratio is maintained as the width is increased at 1 mm/s. Find the rate of increase in the area of the rectangle when the width is 10 cm.

14 A regular hexagon is enlarged such that the regular hexagonal shape is maintained with the length of each side increasing at 1 cm/minute. Find the rate of increase in the area of the hexagon at the instant when the length of each side is 20 cm.

15 Find a formula for the rate of increase in the volume of a sphere when the radius of the sphere, r cm, is increasing at a constant rate of 0.1 cm/s.

a What is the rate of change of volume when $r = 5$?

b What is the radius of the sphere when the rate of change of the volume of the sphere is 40π cm^3/s?

16 A cube is being increased in size such that the length of each edge is increasing at 0.1 cm/s. Find the rate at which **a** the surface area,

b the volume,

of the cube is increasing when the side length is 10 cm.

17 Experts monitoring an oil spill from a crippled oil tanker model the oil slick as a circular disc of radius r metres and thickness 5 cm.

The tanker spills oil into this slick at a rate of 5 m^3/ min. If the thickness of the oil remains at 5 cm find the rate of change in the radius of the slick, in cm/min (correct to nearest cm), when this radius is **a** 20 metres,

b 40 metres,

c 100 metres.

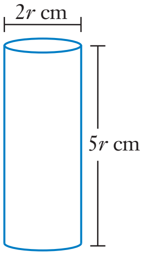
AAP Image/AP/John Gaps III

18 A closed right circular cylinder has base radius r cm and height $5r$ cm.

If r is increasing at $\frac{2}{\pi}$ mm/s, find expressions in terms of r for

a the rate of change in the volume of the cylinder,

b the rate of change in the total external surface area of the cylinder.



19 Oil drips onto a surface and forms a circular film of increasing radius. The oil drips onto the surface at a rate of 1 cm^3/s. Considering the circular film to be of thickness 0.02 cm, find the rate of change in the radius of the film at the instant when this radius is

a 5 cm, **b** 10 cm.

(Give answers in cm/s and correct to one decimal place.)

20 A particle moves in a straight line such that its velocity, v m/s, depends upon displacement, x m, from some fixed point O according to the rule

$$v = 2x^2 - 3$$

Find **a** an expression in terms of x for the acceleration of the particle,

b the velocity and acceleration of the particle when $x = 2$.

21 At what rate is the area of an equilateral triangle increasing when each side is of length 20 cm and each is increasing at a rate of 0.2 cm/s?

22 A large advertising balloon is initially flat and has air pumped into it at a constant rate of 0.5 m^3/s, causing it to adopt a spherical shape of increasing radius.

a Find the rate of increase in the radius of the balloon, to the nearest cm/s, when this radius is

i 1 metre, **ii** 2 metres.

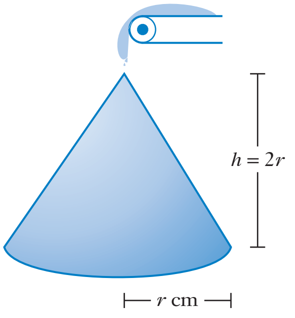
b Find the rate of increase in the radius of the balloon, to the nearest mm/s, twenty seconds after inflation commences.

23 The diagram shows a conical pile of sand with vertical height equal to twice the radius of the base.

Sand is being added to the pile at a rate of 0.25 m^3/minute with the ratio between height and base radius being approximately maintained.

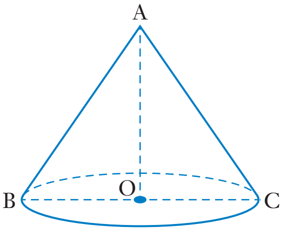
Find **a** the rate of change in the base radius of the cone, when this radius is 2 m.

b the rate of change in the perpendicular height of the cone when this height is 2 m.



24 The cone shown on the right is such that the vertical section through the vertex is an equilateral triangle ABC.

The volume of the cone is increased at a constant rate of V cm^3/s with the equilateral nature of the triangular section maintained. When the base radius is 20 cm the rate of change of this radius is 0.5 cm/s. Find V correct to three significant figures.

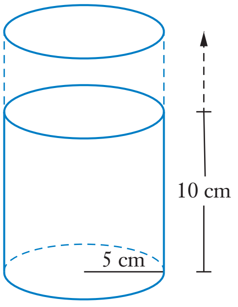


25 A closed cylindrical can of constant base radius 5 cm, has its height increased at 0.1 cm/s, from an initial 10 cm.

Find the rate of change of **a** the external surface area,

and **b** the capacity of the can,

20 seconds after the height increase commenced.



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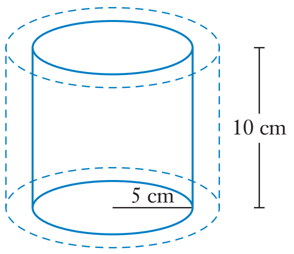
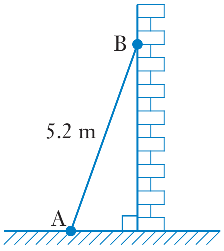
26 A closed cylindrical can of constant height 10 cm, has its base radius increased at 0.1 cm/s, from an initial 5 cm. Find the rate of change of **a** the external surface area, and **b** the capacity of the can, 20 seconds after the radius increase commenced.



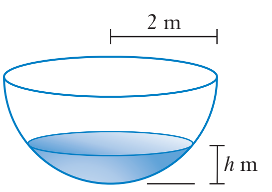
27 The diagram shows a ladder AB, of length 5.2 metres, with its foot on horizontal ground and its top leaning against a vertical wall.

End A slips along the floor, directly away from the wall, at a speed of 0.1 m/s. How fast is end B moving down the wall at the instant that it passes through the point that is 4.8 m above the ground?



28 The diagram shows a hemispherical container of radius 2 metres. When the depth of the liquid in the container is h metres (see diagram), the volume of the liquid, V m^3, is given by $V = \frac{\pi h^2}{3}(6 - h)$.

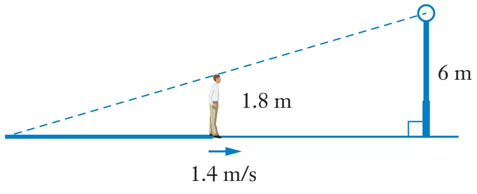
If the liquid is being pumped out of this container at a constant rate of 0.25 m^3/min, find the rate at which h is falling at the instant that $h = 1$. (Answer to nearest mm/min.)



29 The diagram shows a person of height 1.8 m walking towards a lamp post at 1.4 m/s. The light on the lamp post is 6 m above the ground.

a At what rate is the length of the person's shadow changing?

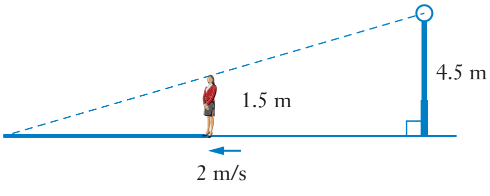
b At what speed is the tip of the person's shadow moving across the ground?



30 The diagram shows a person of height 1.5 m walking away from a lamp post at 2 m/s. The light on the lamp post is 4.5 m above the ground.

a At what rate is the length of the person's shadow changing?

b At what speed is the tip of the person's shadow moving across the ground?



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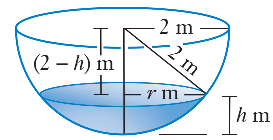
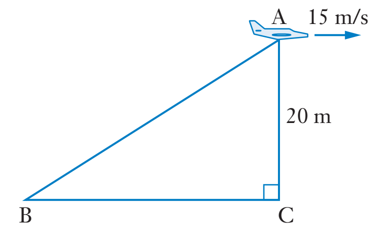
31 The diagram shows a hemispherical container of radius 2 metres. Water is in the container to a depth of h metres (see diagram). Water is removed from the container such that the depth of the water remaining is falling at a constant rate of 0.5 cm/s. Find the rate of change of the radius of the water surface at the instant when the depth is 1 m.



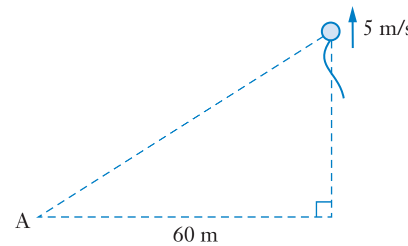
32 The diagram shows a model aeroplane, A, being flown in a straight line, at a constant speed of 15 m/s and constant height 20 m. The plane is radio-controlled by an enthusiast situated at point B (see diagram).

Find the rate at which the distance AB is increasing at the instant that BC (see diagram) is 48 m.



33 The diagram shows a loose balloon rising vertically at a constant 5 m/s, and being observed from point A on the ground, 60 metres from the point on the ground from which the balloon was released.

Find the rate of change in the distance from A to the balloon at the instant that the balloon is at a height of 80 metres.



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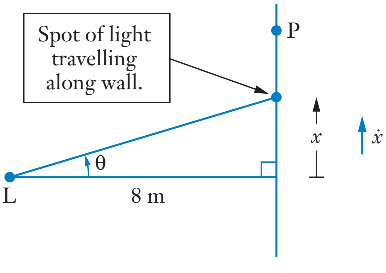
34 In the diagram, point L is the location of a rotating spotlight situated on the top of a police car. L is 8 metres from a straight wall and the light rotates at 2 revolutions per second (i.e. $\frac{d\theta}{dt} = 4\pi$ rad/s).

Find the speed with which the spot from the light is moving along the wall as it passes through point P which is 5 metres from the point on the wall that is closest to the light.



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Small changes

A concept that you should be familiar with from your study of Unit Three of *Mathematics Methods*, and briefly revised here, is that if δx, the change in x, is 'small' then, δy, the corresponding small change in y is given by the **small changes** or **incremental** formula:

$$\frac{\delta y}{\delta x} \approx \frac{dy}{dx} \qquad \text{i.e.} \qquad \delta y \approx \frac{dy}{dx}\delta x.$$

Note

The symbol δ is a Greek letter pronounced *delta*. The capital of this letter is written Δ.
In this calculus context δx is sometimes written as Δx.

EXAMPLE 10

If $f(x) = 2x^3$ use differentiation to find the approximate change in the value of the function when x changes from 10 to 10.1.

Solution

If $\quad y = 2x^3 \quad$ then $\quad \dfrac{dy}{dx} = 6x^2 \quad$ and so $\quad \dfrac{\delta y}{\delta x} \approx 6x^2$

I.e. $\quad \delta y \approx 6x^2\, \delta x$

In this case $x = 10$ and $\delta x = 0.1$, thus $\quad \delta y \approx 6 \times 10^2 \times 0.1 = 60$

When x changes from 10 to 10.1 the change in $f(x)$ is approximately 60.

(Comparing this with $\quad f(10.1) - f(10) = 2 \times 10.1^3 - 2 \times 10^3$
$= 60.602$ shows the approximation reasonable.)

Marginal rates of change

For a company manufacturing x units of a particular commodity there are three important functions of x that will interest the firm.

- The cost function, $C(x)$. This is the total cost of producing x units of the commodity.
- The revenue function, $R(x)$. The total income the firm receives by selling x units.
- The profit function, $P(x)$. The total profit from the sale of x units. $P(x) = R(x) - C(x)$.

(Clearly $R(x)$, $C(x)$ and $P(x)$ only have meaning for $x \geq 0$ and for many commodities it is also the case that these functions only make sense for integer values of x.)

If δx is some small change in x and δC is the corresponding small change in the cost function then $\dfrac{dC}{dx} \approx \dfrac{\delta C}{\delta x}$. Thus if $\delta x = 1$, then $\quad \dfrac{dC}{dx} \approx \dfrac{\delta C}{1}$.

Hence $\dfrac{dC}{dx}$ is the approximate change in C when x changes by 1.

$\dfrac{dC}{dx}$ is called the **marginal cost**. Similarly $\dfrac{dR}{dx}$ **marginal revenue** and $\dfrac{dP}{dx}$ **marginal profit**.

The marginal cost, $C'(x)$, gives us the instantaneous cost per unit when the xth unit is being produced.

This is *not* the average cost per unit at that stage of the production. The average cost per unit can be obtained by dividing the total cost by the number of units, i.e. $\dfrac{C(x)}{x}$.

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EXAMPLE 11

A manufacturer produces x items of a certain product. The cost function $C(x)$ is given in dollars by:

$$C(x) = \frac{x^3}{10} - 30x^2 + 3000x + 5000$$

Evaluate $C'(160)$ and explain what information your answer gives.

Solution

Marginal cost $= \frac{3x^2}{10} - 60x + 3000 \quad \therefore \quad C'(160) = \$1080$ per unit.

When the production level reaches 160 units it will then cost approximately \$1080 to produce one more unit.

Exercise 8D

1 If $f(x) = x^3 - 5x$ use differentiation to find the approximate change in the value of the function when x changes from 5 to 5.01.
Compare your answer to $f(5.01) - f(5)$.

2 If $f(x) = \sin 3x$ use differentiation to find the approximate change in the value of the function when x changes from $\frac{\pi}{9}$ to $\frac{\pi}{9} + 0.01$. Compare your answer to $f\left(\frac{\pi}{9} + 0.01\right) - f\left(\frac{\pi}{9}\right)$.

3 If $f(x) = 2\sin^3 5x$ use differentiation to find the approximate change in the value of the function when x changes from $\frac{\pi}{3}$ to $\frac{\pi}{3} + 0.001$. Compare your answer to $f\left(\frac{\pi}{3} + 0.001\right) - f\left(\frac{\pi}{3}\right)$.

4 The cost function, \$$C$, for producing x units of a product is given by

$$C = 5000 + 20\sqrt{x}.$$

Determine an expression for the marginal cost and hence calculate the marginal cost for

a $x = 25$, **b** $x = 100$, **c** $x = 400$.

5 The cost, \$$C$, of producing x tonnes of a particular product is given by

$$C = 15\,000 + 750x - 15x^2 + \frac{x^3}{10}$$

Determine an expression for the marginal cost and hence calculate the marginal cost for

a $x = 30$, **b** $x = 60$, **c** $x = 100$.

6 The cost function for a particular item is given by \$$C$ where $C = 450 + 0.5x^2$ and x is the number of such items produced.
Find the marginal cost for $x = 10$ and explain what this tells you about the cost of producing one more item at this level of production.

7 The length of one edge of a cube was measured as 5 cm when it was in fact 2 mm more than this. Use differentiation to find the approximate error that using the value of 5 cm would cause in

a the surface area of the cube (in cm^2),

b the volume of the cube (in cm^3).

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Logarithmic differentiation

We conclude this chapter on differentiation techniques and applications by considering a technique sometimes referred to as **logarithmic differentiation**. This assumes that through your study of Unit Four of *Mathematics Methods* you have encountered, and are familiar with, the idea of a logarithm, the natural logarithm $\ln x$, the laws of logarithms and that $\frac{d}{dx}(\ln f(x)) = \frac{f'(x)}{f(x)}$. If this is not the case then leave this section for now and return to it when you have covered these aspects in Unit Four of *Mathematics Methods*.

In order to differentiate some functions it can be useful to consider the logarithm of the expression to be differentiated. Taking logarithms, usually natural logarithms, gives a function defined implicitly, which we can then differentiate. The next example demonstrates its use.

EXAMPLE 12

Determine $\frac{dy}{dx}$ given that $y = 3^x$.

Solution

If $y = 3^x$ then

$$\begin{aligned} \ln y &= \ln(3^x) \\ &= x \ln 3 \end{aligned}$$

Differentiating with respect to x:

$$\frac{d}{dx}(\ln y) = \frac{d}{dx}(x \ln 3)$$

$\therefore$

$$\frac{d}{dy}(\ln y)\frac{dy}{dx} = \frac{d}{dx}(x \ln 3)$$

I.e.

$$\frac{1}{y}\frac{dy}{dx} = \ln 3$$

$$\frac{dy}{dx} = y \ln 3$$

Therefore

$$\frac{dy}{dx} = 3^x \ln 3$$

Exercise 8E

1 Show that using logarithmic differentiation to determine $\frac{dy}{dx}$ for $y = x^3(2x + 1)^5$ gives the same answer as using the product rule.

2 Show that using logarithmic differentiation to determine $\frac{dy}{dx}$ for $y = \frac{x^3}{x^2 + 1}$ gives the same answer as using the quotient rule.

3 Use logarithmic differentiation to differentiate

a x^x **b** x^{2x} **c** $x^{\cos x}$ **d** $\sqrt{\frac{3x+1}{3x-1}}$

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Miscellaneous exercise eight

This miscellaneous exercise may include questions involving the work of this chapter and the ideas mentioned in the Preliminary work section at the beginning of this unit.

1 Find expressions for $\frac{dy}{dx}$ for each of the following.

a $y = \frac{2x+1}{3-2x}$

b $y = \sin^3(2x+1)$

c $3x^2y + y^3 = 5x + 7$

d $x = t^2 + 3t - 6, y = t^4 + 1$

2 Find the equation of the tangent to $x^2 + y^2 = 25$ at the point (3, 4).

3 Find an expression for $\frac{d^2y}{dx^2}$ in terms of y only, given that

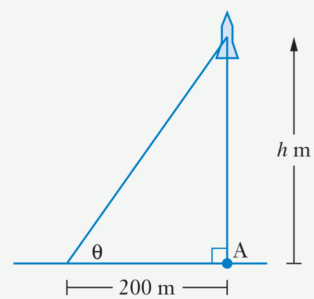
a $y + 1 = xy$.

b $y^3 - 5 = xy$.

4 A rocket is launched vertically from point A, see diagram.

From a point 200 metres from A, and on the same level as A, the rocket's angle of elevation, θ, is recorded.

For the first 25 seconds from launching, the propellant in the rocket is such that the rate of change of θ with respect to time is $\frac{1}{20}$ rad/s.



Find the velocity and acceleration of the rocket when θ is

a $\frac{\pi}{6}$

b $\frac{\pi}{3}$.

5 **a** If $y = (2x+3)^3$ then $x = \frac{\sqrt[3]{y} - 3}{2}$.

Use the first equation to determine $\frac{d^2y}{dx^2}$ and the second to determine $\frac{d^2x}{dy^2}$.

Hence show that $$\frac{d^2y}{dx^2} = -\frac{d^2x}{dy^2}\left(\frac{dy}{dx}\right)^3.$$

b Prove that for any $y = f(x)$, provided the necessary derivatives exist,

$$\frac{d^2y}{dx^2} = -\frac{d^2x}{dy^2}\left(\frac{dy}{dx}\right)^3.$$

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